Chapter 3

INTEGRATION

Note: As was the case in the last chapter, this material is for your edification only! You will not be directly tested on its contents.

A.) The Integral:

1.) Preliminaries:

a.) *Velocity* is defined as a body's *time-rate-of-change of position*. Mathematically, it is written as:

$$v(t) = \frac{d[x(t)]}{dt}.$$

b.) Consider the VELOCITY VERSUS TIME graph shown in Figure 3.1.

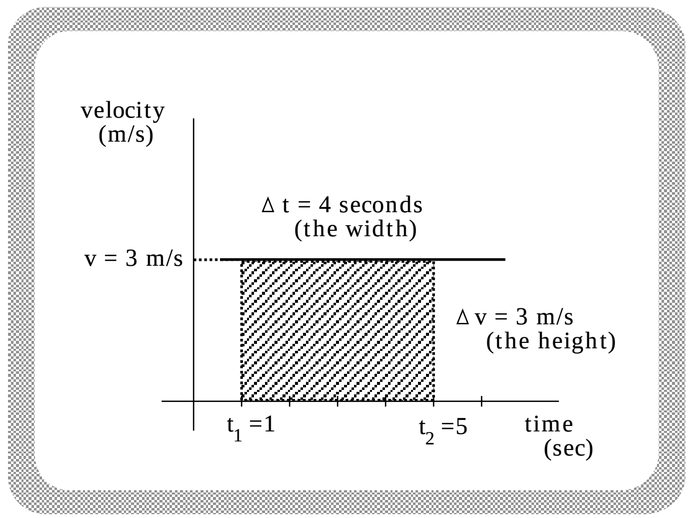

FIGURE 3.1

i.) The distance Δx traveled during the time interval between t_1 and t_2 is $v\,\Delta t = (3\ m/s)(4\ s) = 12\ m$ (this shouldn't be surprising: it is the old "*distance* equals *rate* times *time*" formula you learned when you were a wee one).

ii.) Notice: the distance traveled (i.e., the change of position Δx) is equal to the AREA under the *VELOCITY vs. TIME* curve.

How so? The height of the rectangle is $v = 3\ m/s$ while the width is $\Delta t = 4$ *seconds*. As the area of a rectangle is its *height times width*, the area is $v\,\Delta t$ (or Δx).

2.) Assume a body's *position function* does not change in a linear way (i.e., a function whose derivative is not a constant). How do we determine the body's *net displacement* during a large time interval (i.e., this will be the area under the *VELOCITY vs. TIME* curve during that interval)?

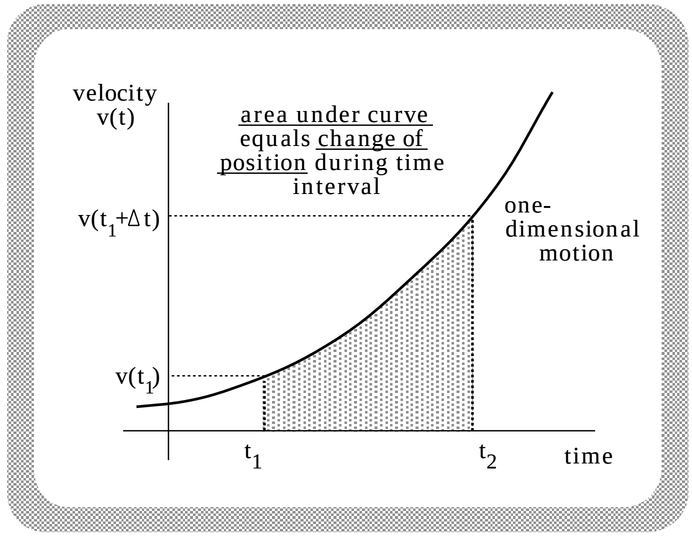

FIGURE 3.2

a.) Assume one-dimensional motion and a *VELOCITY vs. TIME* graph as shown in Figure 3.2.

b.) The distance traveled during the time interval between t_1 and t_2 equals the shaded area under the curve. As the function varies continuously, determining this area is not as easy as was the area-calculation in the previous problem. To accommodate:

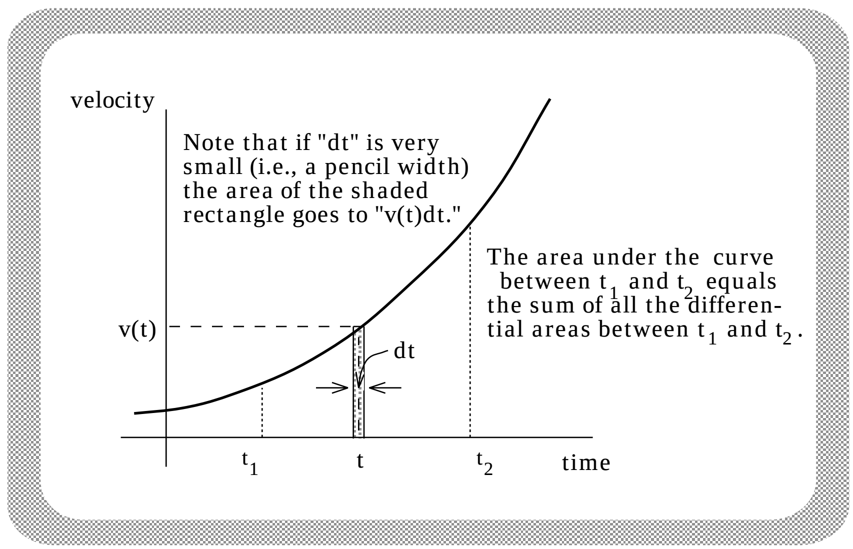

FIGURE 3.3

c.) Consider an arbitrary time *t*:

i.) Place a differential time interval *dt* about time *t* (see Figure 3.3).

Note: This interval should be no thicker than a pencil lead (if that). It has been expanded in the sketch for ease of viewing.

ii.) Call the differential area under the curve bounded by *dt* the *differential displacement dx.* We want to calculate that area (hence, determine that displacement).

iii.) The *differential area* under the curve over the interval *dt* is approximately that of a rectangle (this approximation is made clear in Figure 3.4 to the right). Its area (i.e., *dx*) equals *v(t)dt*. By definition, *v(t) = dx/dt*. As such:

$$dx = [v(t)]\ dt = \left[\frac{dx(t)}{dt}\right]dt.$$

iv.) In English, this equation reads: the *differential displacement dx* of the body equals *the rate at which its position changes with time* (i.e., *dx(t)/dt*) times *the time interval "dt" over which the change occurs.*

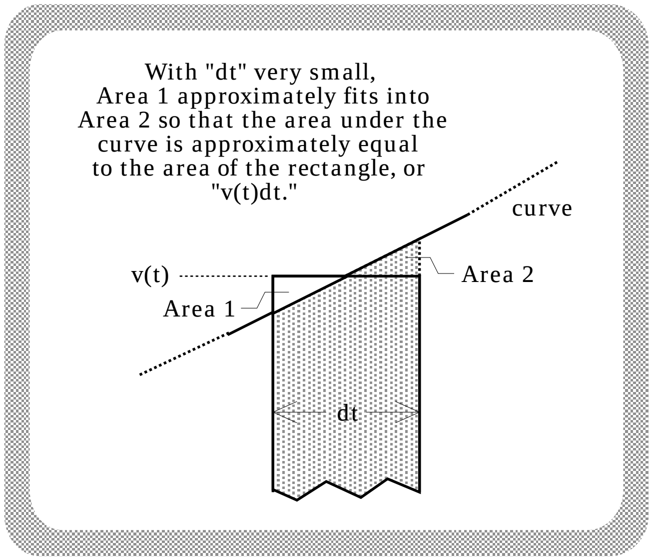

FIGURE 3.4

Note: Because we chose *t* to be any arbitrary time, our expression is good for *any* time.

d.) If we executed this area-finding process for all *dt's* between times t_1 and t_2, then added them all up, we end up with the *total displacement of the body* (i.e., its net *change of position*) between times t_1 and t_2.

3.) The temptation is to use a conventional summation sign "∑" to write out this information in mathematical form. The problem? A "∑" sign is used to denote the summing of *discrete quantities*, not to show the summing of a *continuously varying function.*

4.) When a continuous function is summed, a different sign is used. Called an *integral,* the symbol looks like "∫". With it, we can write:

a.) The sum of the *differential displacements* (i.e., the sum of all the individual *dx's*) between t_1 and t_2 equals the net displacement $x(t_2) - x(t_1)$, or:

$$\int dx = [x(t_2) - x(t_1)].$$

b.) The sum of the *differential displacements dx* between t_1 and t_2 equals the *rate of change of position with time dx/dt* times the *differential time interval dt*, all summed over the time interval between t_1 and t_2. That is:

$$\int dx = \int_{t_1}^{t_2} \left[\frac{dx(t)}{dt}\right] dt .$$

Note: Notice how the limits are placed in the integral.

c.) Putting *Parts a* and *b* together, we can write:

$$\begin{aligned}\int_{t_1}^{t_2} \left[\frac{dx(t)}{dt}\right] dt &= \left[x(t_2) - x(t_1)\right] \\ &= \Delta\left[x(t)\right].\end{aligned}$$

d.) The function *dx(t)/dt* and the function *x(t)* are obviously related to one another (one is the derivative of the other). What is interesting is that *both functions* have shown up in our final equation. Specifically:

i.) We are GIVEN the *time-rate-of-change of the position function* (i.e., *dx/dt*)--that is the quantity inside the integral on the left side of the equal sign.

ii.) We are LOOKING FOR the *position function x(t)* the change Δx of which is denoted on the right side of the equal sign.

5.) Bottom Line: When you find yourselves with an integral like

$$\int_{t=2}^{7} (2t)dt = \ ?$$

you are really looking for a new function (call it *x(t)*) whose *rate of change* (i.e., whose *derivative*) is the function inside the integral. This new function *x(t)* is called *the anti-derivative* of the function inside the integral.

a.) In the context of our example, it means we are looking for a function the derivative of which is *2t* (that function happens to be $x(t) = t^2$).

b.) Once we know that anti-derivative (i.e., $x(t)$), we can evaluate it at the limits to determine how its value *changes* over that time interval (i.e., we can determine $\Delta x(t)$).

c.) To evaluate a function, the upper limit ($t = 7$ *seconds*) is put into the function (i.e., $x(t) = t^2 = (7)^2 = 49$) to determine the function's value at time $t = 7$ *seconds*. From that the evaluation of the function at $t = 2$ *seconds* is subtracted. In that way, the *change of the function over the time interval*--in this case, the net distance traveled between $t = 2$ *seconds* and $t = 7$ *seconds*--is determined.

d.) Writing this out formally, we get:

$$\begin{aligned}\int_{t=2}^{7} (2t)dt &= (t^2)\Big|_{t=2}^{7} \\ &= (7^2) - (2^2) \\ &= 45.\end{aligned}$$

Note: The limits about which the integral's solution is to be evaluated (in this case, $t = 2$ and $t = 7$ *seconds*) are placed as shown above. Occasionally, a bracket notation "$\left[t^2\right]_{t=2}^{7}$" may be used instead.

B.) Conclusions and an Example:

1.) *Integration* allows us to determine the AREA UNDER A CURVE over some specified interval. The solution to an integral is a function *the derivative of which we know*. That is, if we ignore the *dt* (or *dx* or whatever the differential variable happens to be), everything else under the *integral sign* is the derivative of the function we are trying to determine.

2.) Because mathematicians don't attribute physical significance to *differential variables* like *dt* or *dx*, those variables have no meaning in an integral aside from specifying the variable over which the integral summation is to be carried out. By attributing physical significance to such variables, it is possible to think through problems that might otherwise be obscure.

3.) A simple example of this kind of thinking:

a.) Assume we know the circumference of a circle is $2\pi r$, where r is the circle's radius. How can we derive an expression for the *area* of a circle whose radius is R?

b.) Begin by taking a differential HOOP of radius *r* and differential thickness *dr*. A sketch of the hoop is shown in Figure 3.5 to the right.

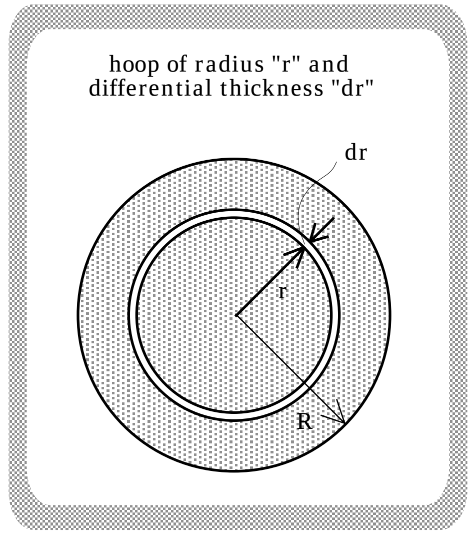

FIGURE 3.5

Note 1: To make it viewable, the differential thickness *dr* has been rendered considerably larger than it actually is.

Note 2: Why take *dr* instead of *dx* or whatever? The variable *r* is usually used when referring to the *radial direction* in POLAR and/or SPHERICAL POLAR notation. Following the convention, the *radial distance* from the circle's center to the arbitrarily chosen hoop is called *r*, and the *differential thickness* of the hoop follows as *dr*.

c.) If we can determine the *differential area* of one arbitrarily chosen hoop, we could do the calculation for all possible differential hoops. Having done so, we could then sum up all those little differential areas to find the *total area* of the circle.

Put mathematically, if we can derive an expression for the *differential* (read this, "small in comparison to the whole") *area dA* of our *hoop*, integrate that expression to sum over all possible hoop areas, then evaluate that integral's solution between $r = 0$ to $r = R$, we will end up with an expression for the total area of the circle.

d.) Executing that operation:

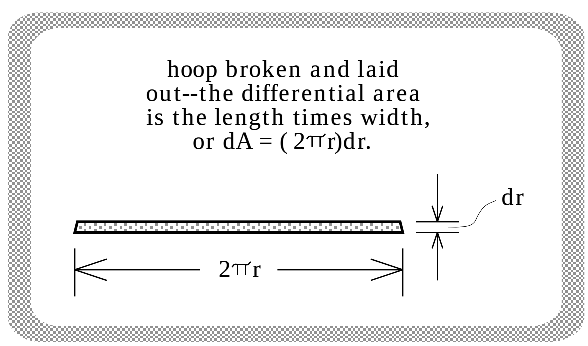

FIGURE 3.6

i.) If *dr* is tiny, the *differential area* of the hoop is equal to the *circumference of the hoop* $(2\pi r)$ times the hoop's thickness *dr* (see Figure 3.6 to the right).

That is, $dA = (2\pi r)dr$.

ii.) Summing yields:

$$
\begin{aligned}
A = \int dA &= \int_{r=0}^{R} (2\pi r)\,dr \\
&= 2\pi \int_{r=0}^{R} (r)\,dr \\
&= 2\pi \left[\frac{r^2}{2} \right]_{r=0}^{R} \\
&= 2\pi \left[\left(\frac{R^2}{2} \right) - (0) \right] \\
&= \pi R^2 .
\end{aligned}
$$

What fun!

QUESTIONS

As was the case with Chapter 2, there are no questions for this section. If you understand this chapter's material, fine. If not, don't panic. It will be a while before we do anything too exotic with integrals.

Note 1: The most important point in this chapter is the idea that problems can be set up by attributing physical significance to differential quantities like *dx* and *dt*. Approaching problems in this way allows one to create *differential equations* that physically reflect the situations one is trying to model.

Note 2: Approaching Calculus with the idea that the *dt's* and *dx's* have significance is particularly useful when one is trying to make sense of a differential equation. An astute reader can find within such an expression meaning that is not at all obvious within a so-called *pure math* context.